

Instanton Representation of Plebanski Gravity. Consistency of the Initial Value Constraints under Time Evolution

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Abstract: The instanton representation of Plebanski gravity provides as equations of motion a Hodge self-duality condition and a set of “generalized” Maxwell’s equations, subject to gravitational degrees of freedom encoded in the initial value constraints of General Relativity. The main result of the present paper will be to prove that this constraint surface is preserved under time evolution. We carry this out not using the usual Dirac procedure, but rather the Lagrangian equations of motion themselves. Finally, we provide a comparison with the Ashtekar formulation to place these results into overall context.

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§1. Introduction. In [1] a new formulation of General Relativity has been presented, referred to as the instanton representation of Plebanski gravity. The basic dynamical variables are an $SO(3, C)$ gauge connection A_μ^a and a matrix Ψ_{ae} taking its values in two copies of $SO(3, C)$.[†] The consequences of the associated action I_{Inst} were determined via its equa-

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[†]Index labelling conventions for this paper are that symbols a, b, \dots from the beginning of the Latin alphabet denote internal $SO(3, C)$ indices while those from the middle i, j, k, \dots denote spatial indices. Both of these sets of indices take values 1, 2 and 3. The Greek symbols μ, ν, \dots refer to spacetime indices which take values 0, 1, 2, 3.

tions of motion, which hinge crucially on weak equalities implied by the initial value constraints. For these consequences to be self-consistent, the constraint surface must be preserved for all time by the evolution equations. The present paper will show that this is indeed the case. Due to the necessity to avoid some technical difficulties, we will not use the usual Dirac formulation for totally constrained systems [2]. In fact we will not make use of Poisson brackets or of any canonical structure implied by I_{Inst} . Rather, we will deduce the time evolution of the initial value constraints directly from the Lagrangian equations of motion of I_{Inst} .

The organization of this paper is as follows. §2 provides some background on the relation between I_{Inst} and the Ashtekar formulation. There is a common notion that these theories are the same within their common domain of definition. §2 argues that this is not the case, which sets the stage for the present paper. §3 and §4 present I_{Inst} as a stand-alone action and derive the time evolution of the basic variables. §5, §6 and §7 demonstrate that the nondynamical equations, referred to as the diffeomorphism, Gauss' law and Hamiltonian constraints, evolve into combinations of the same constraint set. The result is that the time derivatives of these constraints are weakly equal to zero with no additional constraints generated on the system. While we do not use the usual Dirac procedure in this paper, the result is still that I_{Inst} is in a sense Dirac-consistent. We will make this inference clearer by comparison with the Ashtekar formulation in §8. On a final note, the terms *diffeomorphism* and *Gauss' law* constraints are used loosely in this paper, in that we have not specified what transformations of the basic variables these constraints generate. The use of these terms will be primarily for notational purposes, due to their counterparts which appear in the Ashtekar formalism.

§2. Background: Relation of the instanton representation to the Ashtekar formalism. The action for the instanton representation can be written in the following 3+1 decomposed form [1]

$$I_{\text{Inst}} = \int dt \int_{\Sigma} d^3x \left[\Psi_{ae} B_e^i \dot{A}_i^a + A_0^a B_e^i D_i \{ \Psi_{ae} \} - \epsilon_{ijk} N^i B_a^j B_e^k \Psi_{ae} - \right. \\ \left. - iN \sqrt{\det \| B \|} \sqrt{\det \| \Psi \|} (\Lambda + \text{tr } \Psi^{-1}) \right], \quad (1)$$

where D_i is the $\text{SO}(3, \mathbb{C})$ covariant derivative, whose action on $\text{SO}(3, \mathbb{C})$ -valued 3-vectors v_a is given by

$$D_i v_a = \partial_i v_a + f_{abc} A_i^b v_c \quad (2)$$

with structure constants $f_{abc} = \epsilon_{abc}$. The phase space variables are a spatial $\text{SO}(3, \mathbb{C})$ connection A_i^a with magnetic field B_a^i and a matrix $\Psi_{ae} \in \text{SO}(3, \mathbb{C}) \otimes \text{SO}(3, \mathbb{C})$, and the quantities (A_0^a, N, N^i) are nondynamical fields. One would like to compute the Hamiltonian dynamics of (1) using phase space variables $\Omega_{\text{Inst}} = (\Psi_{ae}, A_i^a)$ as the fundamental fields. But the phase space of (1) is noncanonical since its symplectic two form,

$$\begin{aligned} \Omega_{\text{Inst}} &= \delta\theta_{\text{Inst}} = \delta \left(\int_{\Sigma} d^3x \Psi_{ae} B_e^i \delta A_i^a \right) = \\ &= \int_{\Sigma} d^3x B_e^i \delta \Psi_{ae} \wedge \delta A_i^a + \int_{\Sigma} d^3x \Psi_{ae} \epsilon^{ijk} D_j (\delta A_k^e) \wedge \delta A_i^a, \end{aligned} \quad (3)$$

is not closed owing to the presence of the second term on the right hand side of (3). The equations of motion for (A_0^a, N, N^i) define a constraint surface on Ω_{Inst} , which as a necessary condition for self-consistency must be shown to be preserved under time evolution.

The initial stages of the Dirac procedure for constrained systems [2] applied to (1) imply that the momentum canonically conjugate to A_i^a yields the primary constraint

$$\Pi_a^i = \frac{\delta I_{\text{Inst}}}{\delta \dot{A}_i^a} = \Psi_{ae} B_e^i, \quad (4)$$

where $\det \|B\|$ and $\det \|\Psi\|$ are nonzero. Then making the identification $\tilde{\sigma}_a^i = \Pi_a^i$ and upon substitution of (4) into (1), one obtains the action

$$I_{\text{Ash}} = \int dt \int_{\Sigma} d^3x \left[\tilde{\sigma}_a^i \dot{A}_i^a + A_0^a G_a - N^i H_i - \frac{i}{2} \underline{N} H \right], \quad (5)$$

where (G_a, N^i, N) are the Gauss' law, vector and Hamiltonian constraints given by

$$G_a = D_i \tilde{\sigma}_a^i, \quad H_i = \epsilon_{ijk} \tilde{\sigma}_a^j B_a^k, \quad H = \epsilon_{ijk} \epsilon^{abc} \tilde{\sigma}_a^i \tilde{\sigma}_b^j \left(\frac{\Lambda}{3} \tilde{\sigma}_c^k + B_c^k \right). \quad (6)$$

Equation (5) is the action for the Ashtekar formulation of General Relativity [3, 4] defined on the phase space $\Omega_{\text{Ash}} = (\tilde{\sigma}_a^i, A_i^a)$, where $\tilde{\sigma}_a^i$ is the densitized triad. The auxiliary fields $(A_0^a, N^i, \underline{N})$ are $\text{SO}(3, \mathbb{C})$ rotation angle A_0^a , the shift vector N^i and the densitized lapse function $\underline{N} = N(\det \|\tilde{\sigma}\|)^{-1/2}$. From (5) one reads off the symplectic two form Ω_{Ash} given by

$$\Omega_{\text{Ash}} = \int_{\Sigma} d^3x \delta \tilde{\sigma}_a^i \wedge \delta A_i^a = \delta \left(\int_{\Sigma} d^3x \tilde{\sigma}_a^i \delta A_i^a \right) = \delta \theta_{\text{Ash}}, \quad (7)$$

which is the exact functional variation of the canonical one form θ_{Ash} .

This implies the following Poisson brackets between any two phase space functions f and g for fundamental variables Ω_{Ash} defined on three-dimensional spatial hypersurfaces Σ

$$\{f, g\} = \int_{\Sigma} d^3x \left(\frac{\delta f}{\delta \tilde{\sigma}_a^i(x)} \frac{\delta g}{\delta A_i^a(x)} - \frac{\delta g}{\delta \tilde{\sigma}_a^i(x)} \frac{\delta f}{\delta A_i^a(x)} \right). \quad (8)$$

Since equation (8) is of canonical form, it is straightforward to compute the constraints algebra and the Hamilton's equations of motion for (5). The constraints algebra for (6) based on these Poisson brackets is

$$\left. \begin{aligned} \{\vec{H}[\vec{N}], \vec{H}[\vec{M}]\} &= H_k [N^i \partial^k M_i - M^i \partial^k N_i] \\ \{\vec{H}[N], G_a[\theta^a]\} &= G_a [N^i \partial_i \theta^a] \\ \{G_a[\theta^a], G_b[\lambda^b]\} &= G_a [f_{bc}^a \theta^b \lambda^c] \\ \{H(\underline{N}), \vec{H}[\vec{N}]\} &= H [N^i \partial_i \underline{N}] \\ \{H(\underline{N}), G_a(\theta^a)\} &= 0 \\ [H(\underline{N}), H(\underline{M})] &= H_i [(N \partial_j \underline{M} - \underline{M} \partial_j N) H^{ij}] \end{aligned} \right\}, \quad (9)$$

with structure functions $H^{ij} = \tilde{\sigma}_a^i \tilde{\sigma}_a^j$, which is first class due to closure. Therefore the algebra (9) is consistent in the Dirac sense.

Following the step-by-step Dirac procedure, one would be led naively to the conclusion that (1), shown in [1] to describe General Relativity for certain Petrov types, for $\det\|B\| \neq 0$ and $\det\|\Psi\| \neq 0$ is the same theory as (5) which also describes General Relativity. One might then infer, on account of (4), the Dirac-consistency of (1). In this paper we will probe beyond the surface and show that (1) and (5) are indeed different versions of General Relativity. Certainly as a minimum, one can regard (1) as a noncanonical version of (5) which is canonical.

As a first step via the standard Hamiltonian approach, one should compute the Hamiltonian dynamics of (1) using Poisson brackets constructed from the inverse of the symplectic matrix derivable from (3), without making use of (4). However the implementation of these Poisson brackets in practice presently appears to be unclear, and will require some additional research.* To substantiate the claim that (1) is

*The fundamental Poisson brackets of (1) are noncanonical and have been computed in Appendix A. The present difficulty lies specifically in the interpretation of the sequence of the action of spatial derivatives on the phase space variables when one considers the full theory. We will therefore relegate as a direction of future research the computation of the associated constraints algebra.

at some level fundamentally different from (5) while at the same time being self-consistent, we must therefore find an alternate means for verifying consistency of the constraints defined on $\Omega_{\text{Inst}} = (\Psi_{ae}, A_i^a)$ under time evolution. Our method will be to use the Lagrangian equations of motion of (1) as the starting point. In this way, we will avoid the necessity to define a canonical structure and Poisson brackets for (1), which appear from Appendix A to be relatively complicated.

§3. Instanton representation of Plebanski gravity. After an integration of parts with discarding of boundary terms, using $F_{0i}^a = \dot{A}_i^a - D_i A_0^a$ for the temporal curvature components, the starting action for the instanton representation of Plebanski gravity (1) can be written as [1]

$$I_{\text{Inst}} = \int dt \int_{\Sigma} d^3x \Psi_{ae} B_e^k (F_{0k}^a + \epsilon_{kjm} B_a^j N^m) - iN \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}), \quad (10)$$

where $N^\mu = (N, N^i)$ are the lapse function and shift vector from the metric of General Relativity, and Λ is the cosmological constant. The basic fields are Ψ_{ae} and A_i^a , and the action (10) is defined only on configurations for which $\det\|B\| \neq 0$ and $\det\|\Psi\| \neq 0$.* In the Dirac procedure one refers to N^μ as nondynamical fields, since their velocities do not appear in the action. While the velocity $\dot{\Psi}_{ae}$ also does not appear, it is important to distinguish this field from N^μ since the action (10), unlike the case for N^μ , is nonlinear in Ψ_{ae} .†

The equation of motion for the shift vector N^i , the analogue of Hamilton's equation for its conjugate momentum $\Pi_{\bar{N}}$, is given by

$$\frac{\delta I_{\text{Inst}}}{\delta N^i} = \epsilon_{ijk} B_a^j B_e^k \Psi_{ae} = (\det\|B\|) (B^{-1})_i^d \psi_d \sim 0, \quad (11)$$

where $\psi_d = \epsilon_{dae} \Psi_{ae}$ is the antisymmetric part of Ψ_{ae} . This is equivalent to the diffeomorphism constraint H_i owing to the nondegeneracy of B_a^i , and we will often use H_i and ψ_d interchangeably in this paper. The equation of motion for the lapse function N , the analogue of Hamilton's equation for its conjugate momentum Π_N , is given by

$$\frac{\delta I_{\text{Inst}}}{\delta N} = \sqrt{\det\|B\|} \sqrt{\det\|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}) = 0. \quad (12)$$

*The latter case limits the application of our results to spacetimes of Petrov Types I, D and O (see e.g. [6] and [7]).

†Additionally, since Ψ_{ae} multiplies the velocity of another field, then according to the instanton representation it should accurately be regarded more-so as an intrinsic part of the canonical structure than as a nondynamical field.

Nondegeneracy of Ψ_{ae} and of the magnetic field B_e^i implies that on-shell, the following relation must be satisfied

$$\Lambda + \text{tr} \Psi^{-1} = 0, \quad (13)$$

which we will similarly treat as being synonymous with the Hamiltonian constraint (12). The equation of motion for Ψ_{ae} is

$$\begin{aligned} \frac{\delta I_{\text{Inst}}}{\delta \Psi_{ae}} &= B_e^k F_{0k}^a + \epsilon_{kjm} B_e^k B_a^j N^m + \\ &+ iN \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} (\Psi^{-1} \Psi^{-1})^{ea} \sim 0, \end{aligned} \quad (14)$$

up to a term proportional to (13) which we have set weakly equal to zero. One could attempt to define a momentum conjugate to Ψ_{ae} , for which (14) would be the associated Hamilton's equation of motion. But since Ψ_{ae} forms part of the canonical structure of (10), then our interpretation is that this is not necessary.*

The equation of motion for the connection A_μ^a is given by

$$\begin{aligned} \frac{\delta I_{\text{Inst}}}{\delta A_\mu^a} &\sim \epsilon^{\mu\sigma\nu\rho} D_\sigma (\Psi_{ae} F_{\nu\rho}^e) - \frac{i}{2} \delta_i^\mu \bar{D}_{da}^{ji} \left[4\epsilon_{mjk} N^m B_e^k \Psi_{[de]} + \right. \\ &\left. + N (B^{-1})_j^d \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}) \right] \sim 0, \end{aligned} \quad (15)$$

where we have defined

$$\left. \begin{aligned} \bar{D}_{ea}^{ji}(x, y) &\equiv \frac{\delta B_e^j(y)}{\delta A_i^a(x)} = \epsilon^{jki} (-\delta_{ae} \partial_k + f_{eda} A_k^d) \delta^{(3)}(x, y) \\ \bar{D}_{ea}^{0i} &\equiv 0 \end{aligned} \right\}. \quad (16)$$

The terms in large square brackets in (15) vanish weakly, since they are proportional to the constraints (11) and (13) and their spatial derivatives. Hence we can regard (15) as being synonymous with

$$\epsilon^{\mu\sigma\nu\rho} D_\sigma (\Psi_{ae} F_{\nu\rho}^e) \sim 0. \quad (17)$$

In an abuse of notation, we will treat (14) and (17) as strong equalities in this paper. This will be justified once we have completed the demonstration that the constraint surface defined collectively by (11), (12) and the Gauss's law constraint contained in (17) is indeed preserved under time evolution. As a note prior to proceeding we will often make

*This is because (14) contains a velocity \dot{A}_k^a within F_{0k}^a and will therefore be regarded as an evolution equation rather than a constraint. This is in stark contrast with (11) and (12), which are genuine constraint equations due to the absence of any velocities.

the following identification derived in [1]

$$N\sqrt{\det\|B\|}\sqrt{\det\|\Psi\|} \equiv \sqrt{-g} \quad (18)$$

as a shorthand notation, to avoid cluttering many of the derivations which follow in the present paper.

§3.1. Internal consistency of the equations of motion. Prior to embarking upon the issue of consistency of time evolution of the initial value constraints, we will check for internal consistency of I_{Inst} , which entails probing of the physical content implied by (17) and (14). First, equation (17) can be decomposed into its spatial and temporal parts as

$$D_i(\Psi_{bf}B_f^i) = 0, \quad D_0(\Psi_{bf}B_f^i) = \epsilon^{ijk}D_j(\Psi_{bf}F_{0k}^f). \quad (19)$$

The first equation of (19) is the Gauss' law constraint of a SO(3) Yang–Mills theory, when one makes the identification of $\Psi_{bf}B_f^i \sim E_b^i$ with the Yang–Mills electric field. The Maxwell equations for U(1) gauge theory with sources (ρ, \vec{J}) , in units where $c=1$, are given by

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \dot{B} = -\vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{E} = \rho, \quad \dot{E} = -\vec{J} + \vec{\nabla} \times \vec{B}. \quad (20)$$

Equations (19) can be seen as a generalization of the first two equations of (20) to SO(3) nonabelian gauge theory in flat space when:

- 1) One identifies $F_{0k}^f \equiv E_k^f$ with the SO(3) generalization of the electric field \vec{E} , and
- 2) One chooses $\Psi_{ae} = k\delta_{ae}$ for some numerical constant k .

When $\rho=0$ and $\vec{J}=0$, then one has the vacuum theory and equations (20) are invariant under the transformation

$$(\vec{E}, \vec{B}) \longrightarrow (-\vec{B}, \vec{E}). \quad (21)$$

Then the second pair of equations of (20) become implied by the first pair. This is the condition that the Abelian curvature $F_{\mu\nu}$, where $F_{0i} = E_i$ and $\epsilon_{ijk}F_{jk} = B_i$, is Hodge self-dual with respect to the metric of a conformally flat spacetime. But equations (19) for more general Ψ_{ae} encode gravitational degrees of freedom, which as shown in [1] generalizes the concept of self-duality to more general spacetimes solving the Einstein equations. Let us first attempt to derive the analogue for (19) of the second pair of equations in (20) in the vacuum case. Acting on the first equation of (19) with D_0 yields

$$D_0D_i(\Psi_{bf}B_f^i) = D_iD_0(\Psi_{bf}B_f^i) + [D_0, D_i](\Psi_{bf}B_f^i) = 0. \quad (22)$$

Substituting the second equation of (19) into the first term on the right hand side of (22) and using the definition of temporal curvature as the commutator of covariant derivatives on the second term we have

$$\begin{aligned} D_i \left[\epsilon^{ijk} D_j (\Psi_{bf} F_{0k}^f) \right] + f_{bcd} F_{0i}^c \Psi_{df} B_f^i &= \\ &= f_{bcd} (B_c^k F_{0k}^f + B_f^k F_{0k}^c) \Psi_{df} = 0, \end{aligned} \quad (23)$$

where we have also used the spatial part of the commutator $\epsilon^{ijk} D_i D_j v_a = f_{abc} B_b^k v_c$. Note that the right hand side of (23) is symmetric in f and c , and also forms the symmetric part of the left hand side of (14)

$$B_f^i F_{0i}^b + i\sqrt{-g} (\Psi^{-1} \Psi^{-1})^{fb} + \epsilon_{ijk} B_f^i B_b^j N^k = 0, \quad (24)$$

re-written here for completeness. To make progress from (23), we will substitute (24) into (23). This causes the last term of (24) to drop out due to antisymmetry, which leaves us with

$$\begin{aligned} -i\sqrt{-g} f_{bcd} \left[\Psi_{df} (\Psi^{-1} \Psi^{-1})^{fc} + \Psi_{df} (\Psi^{-1} \Psi^{-1})^{fc} \right] &= \\ &= -2i\sqrt{-g} f_{bcd} \Psi_{dc}^{-1}. \end{aligned} \quad (25)$$

The equations are consistent only if (25) vanishes, which is the requirement that $\Psi_{ae} = \Psi_{ea}$ be symmetric. This of course is the requirement that the diffeomorphism constraint (11) be satisfied. So the analogue of the second pair of (20) in the vacuum case must be encoded within the requirement that $\Psi_{ae} = \Psi_{ea}$ be symmetric.

§4. The time evolution equations. We must now verify that the initial value constraints are preserved under time evolution defined by the equations of motion (14) and (15). Since the temporal part of (19) is already a constraint, then the only equality required is the Hodge duality condition

$$B_f^k F_{0k}^b + i\sqrt{-g} (\Psi^{-1} \Psi^{-1})^{fb} + \epsilon_{ijk} N^i B_b^j B_f^k = 0, \quad (26)$$

and the spatial part of (19)

$$\epsilon^{ijk} D_j (\Psi_{ae} F_{0k}^e) = D_0 (\Psi_{ae} B_e^i). \quad (27)$$

Since the initial value constraints were used to obtain the second line of (26) from (10), then we must verify that these constraints are preserved under time evolution as a requirement of consistency. Using $F_{0i}^b = \dot{A}_i^b - D_i A_0^b$ and defining

$$\sqrt{-g} (B^{-1})_i^f (\Psi^{-1} \Psi^{-1})^{fb} + \epsilon_{mnk} N^m B_b^n \equiv iH_k^b, \quad (28)$$

then equation (26) can be written as a time evolution equation for the connection A_i^a . Note that this is not the same thing as a constraint equation, as noted previously,

$$F_{0i}^b = -iH_i^b \longrightarrow \dot{A}_i^b = D_i A_0^b - iH_i^b. \quad (29)$$

From equation (29) we can obtain the following time evolution equation for the magnetic field B_a^i , given by

$$\begin{aligned} \dot{B}_e^i &= \epsilon^{ijk} D_j \dot{A}_k^e = \epsilon^{ijk} D_j (D_k A_0^e - iH_k^e) = \\ &= f_{ebc} B_b^i A_0^c - i\epsilon^{ijk} D_j H_k^e = -\delta_{\vec{\theta}} B_e^i - i\epsilon^{ijk} D_j H_k^e, \end{aligned} \quad (30)$$

which will be useful. On the first term on the right hand side of (30) we have used the definition of the curvature as the commutator of covariant derivatives. The notation $\delta_{\vec{\theta}}$ in (30) suggests that that B_e^i transforms as a $SO(3, C)$ vector under gauge transformations parametrized by $\theta^b \equiv A_0^b$. Since we have not defined the canonical structure of I_{Inst} , then $\delta_{\vec{\theta}}$ as used in (30) and in (33) should at this stage be regarded simply as a shorthand notation.

We will now apply the Leibnitz rule in conjunction with the definition of the temporal covariant derivatives to (27) to determine the equation governing the time evolution of Ψ_{ae} . This is given by

$$D_0(\Psi_{ae} B_e^i) = B_e^i \dot{\Psi}_{ae} + \Psi_{ae} \dot{B}_e^i + f_{abc} A_0^b (\Psi_{ce} B_e^i) = \epsilon^{ijk} D_j (\Psi_{ae} F_{0k}^e). \quad (31)$$

Substituting (30) and (29) into both sides of (31), we have

$$\begin{aligned} B_e^i \dot{\Psi}_{ae} + \Psi_{ae} (f_{ebc} B_b^i A_0^c - i\epsilon^{ijk} D_j H_k^e) + f_{abc} A_0^b (\Psi_{ce} B_e^i) = \\ = -i\epsilon^{ijk} D_j (\Psi_{ae} H_k^e). \end{aligned} \quad (32)$$

In what follows, it will be convenient to use the following transformation properties for Ψ_{ae} as A_i^a under $SO(3, C)$ gauge transformations*

$$\left. \begin{aligned} \delta_{\vec{\theta}} \Psi_{ae} &= (f_{abc} \Psi_{ce} + f_{ebc} \Psi_{ac}) A_0^b \\ \delta_{\vec{\theta}} A_i^a &= -D_i A_0^a \\ \delta_{\vec{\theta}} B_e^i &= -f_{ebc} B_b^i A_0^c \end{aligned} \right\}. \quad (33)$$

Then using (33), the time evolution equations for the phase space variables Ω_{Inst} can be written in the following compact form

$$\dot{A}_i^b = -\delta_{\vec{\theta}} A_i^b - iH_i^b, \quad \dot{\Psi}_{ae} = -\delta_{\vec{\theta}} \Psi_{ae} - i\epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) H_k^f. \quad (34)$$

*Note that these are based purely on the transformation properties of a $SO(3, C)$ gauge connection and of a second-rank $SO(3, C)$ tensor, which hold irrespective of any canonical formalism.

We have determined the evolution equations for Ψ_{ae} and A_i^a directly from I_{Inst} . Recall that we have not used Poisson brackets, and have assumed that the constraints G_a , H_i and H vanish weakly. Therefore the first order of business will be to check for the preservation of the initial value constraints under the time evolution generated by (34). This means that we must check that the time evolution of the diffeomorphism, Gauss' law and Hamiltonian constraints are combinations of terms proportional to the same set of constraints and their spatial derivatives, and terms which vanish when the constraints hold.*

These constraints are given by

$$\left. \begin{aligned} \mathbf{w}_e \{ \Psi_{ae} \} &= 0 \\ (\det \| B \|) (B^{-1})^d{}_i \psi_d &= 0 \\ \sqrt{\det \| B \|} \sqrt{\det \| \Psi \|} (\Lambda + \text{tr } \Psi^{-1}) &= 0 \end{aligned} \right\}, \quad (35)$$

where $\det \| B \| \neq 0$ and $\det \| \Psi \| \neq 0$. We will occasionally make the identification

$$N \sqrt{\det \| B \|} \sqrt{\det \| \Psi \|} \equiv \sqrt{-g} \quad (36)$$

for a shorthand notation. Additionally, the following definitions are provided for the vector fields appearing in the Gauss constraint

$$\mathbf{w}_e = B_e^i D_i, \quad \mathbf{v}_e = B_e^i \partial_i, \quad (37)$$

where D_i is the $\text{SO}(3, \mathbb{C})$ covariant derivative with respect to the connection A_i^a . Recall that equations (35) are precisely the equations of motion for the auxiliary fields A_0^a , N^i and N in (1).

§5. Consistency of the diffeomorphism constraint under time evolution. The diffeomorphism constraint is directly proportional to $\psi_d = \epsilon_{dae} \Psi_{ae}$, the antisymmetric part of Ψ_{ae} . So to establish the consistency condition for this constraint, it suffices to show that the antisymmetric part of the second equation of (34) vanishes weakly. This is given by

$$\epsilon_{dae} \dot{\Psi}_{ae} = -\delta_{\vec{\theta}} (\epsilon_{dae} \Psi_{ae}) - i \epsilon_{dae} \epsilon^{ijk} (B^{-1})^e{}_i (D_j \Psi_{af}) H_k^f, \quad (38)$$

which splits into two terms. Using (33), one finds that the first term on the right hand side of (38) is given by

$$-\epsilon_{dae} \delta_{\vec{\theta}} \Psi_{ae} = -\epsilon_{dae} (f_{abc} \Psi_{ce} + \Psi_{ac} f_{bc}) A_0^b. \quad (39)$$

*This includes any nonlinear function of linear order or higher in the constraints, a situation which involves the diffeomorphism constraint.

In what follows and in various other places in this paper, we will use the fact that the $SO(3)$ structure constants f_{abc} are numerically the same as the three-dimensional epsilon symbol ϵ_{abc} . So the following identities hold

$$\epsilon_{abc}\epsilon_{fec} = \delta_{af}\delta_{be} - \delta_{ae}\delta_{bf}, \quad \epsilon_{abc}\epsilon_{ebc} = 2\delta_{ae}, \quad \epsilon_{abc}\epsilon_{abc} = 6. \quad (40)$$

Using (40), then (39) is given by

$$\begin{aligned} -\epsilon_{dae}\delta_{\bar{\theta}}\Psi_{ae} &= -\left[(\delta_{eb}\delta_{dc} - \delta_{ec}\delta_{bd})\Psi_{ce} + (\delta_{db}\delta_{ac} - \delta_{dc}\delta_{ab})\Psi_{ac}\right]A_0^b = \\ &= -(\Psi_{db} - \delta_{bd}\text{tr}\Psi + \delta_{db}\text{tr}\Psi - \Psi_{bd})A_0^b = 2\Psi_{[bd]}A_0^b = -\epsilon_{dbh}A_0^b\psi_h, \end{aligned} \quad (41)$$

which is proportional to the diffeomorphism constraint. The second term on the right hand side of (38) has two contributions due to H_k^f as defined in (28), and the first contribution reduces to

$$\begin{aligned} -i\epsilon_{dae}\epsilon^{ijk}(B^{-1})_i^e(D_j\Psi_{af})(H_{(1)})_k^f &= \\ = -i\epsilon_{dae}\epsilon^{ijk}(B^{-1})_i^e(D_j\Psi_{af})\sqrt{-g}(B^{-1})_k^g(\Psi^{-1}\Psi^{-1})^{gf}. \end{aligned} \quad (42)$$

Using the definition of the determinant of nondegenerate 3×3 matrices

$$\epsilon^{ijk}\epsilon_{abc}(B^{-1})_j^b(B^{-1})_k^c = B_a^i(\det\|B\|)^{-1}, \quad (43)$$

then (42) further simplifies to

$$\begin{aligned} i\epsilon_{dae}(\det\|B\|)^{-1}\epsilon^{egh}(\Psi^{-1}\Psi^{-1})^{gf}B_h^jD_j\Psi_{af} &= \\ = i(\det\|B\|)^{-1}(\Psi^{-1}\Psi^{-1})^{gf}(\delta_d^g\delta_a^h - \delta_a^g\delta_d^h)\mathbf{v}_d\{\Psi_{af}\} &= \\ = i(\det\|B\|)^{-1}(\Psi^{-1}\Psi^{-1})^{gf}(\delta_d^g\mathbf{v}_a\{\Psi_{af}\} - \mathbf{v}_d\{\Psi_{gf}\}) &= \\ = i(\det\|B\|)^{-1}\left[(\Psi^{-1}\Psi^{-1})^{df}G_f + \mathbf{v}_d\{\Lambda + \text{tr}\Psi^{-1}\}\right]. \end{aligned} \quad (44)$$

The first term on the final right hand side of (44) is proportional to the Gauss' law constraint and the second term to the derivative of a term proportional to the Hamiltonian constraint.* The second contribution to the second term of (38) is given by

$$\begin{aligned} \epsilon_{dae}\epsilon^{ijk}(B^{-1})_i^e(D_j\Psi_{af})(H_{(2)})_k^f &= \\ = \epsilon_{dae}\epsilon^{ijk}(B^{-1})_i^e(D_j\Psi_{af})\epsilon_{mnk}N^mB_f^n &= \\ = \epsilon_{dac}(\delta_m^i\delta_n^j - \delta_n^i\delta_m^j)(B^{-1})_i^e(D_j\Psi_{af})N^mB_f^n, \end{aligned} \quad (45)$$

*We have added in a term Λ , which can be regarded as a constant of integration with respect to the spatial derivatives from \mathbf{v}_d .

where we have used the analogue of the first identity of (40) for spatial indices. Then (45) further simplifies to

$$\begin{aligned} \epsilon_{dae} N^i (B^{-1})_i^e \mathbf{v}_f \{ \Psi_{af} \} - N^j D_j (\epsilon_{dae} \Psi_{ae}) = \\ = \epsilon_{dae} N^i (B^{-1})_i^e G_a - N^j D_j \psi_d. \end{aligned} \quad (46)$$

The result is that the time evolution of the diffeomorphism constraint is directly proportional to

$$\begin{aligned} \dot{\psi}_d = \left[i (\det \| B \|)^{-1} (\Psi^{-1} \Psi^{-1})^{da} + \epsilon_{dae} N^i (B^{-1})_i^e \right] G_a + \\ + (A_0^b \epsilon_{bdh} - \delta_{dh} N^j D_j) \psi_h + i (\det \| B \|)^{-1} \mathbf{v}_d \{ (-g)^{-1/2} H \}, \end{aligned} \quad (47)$$

which is a linear combination of terms proportional to the constraints (35) and their spatial derivatives. The result is that the diffeomorphism constraint $H_i = 0$ is consistent with respect to the Hamiltonian evolution generated by the equations (34). So it remains to verify consistency of the Gauss' law and the Hamiltonian constraints G_a and H .

§6. Consistency of the Gauss constraint under time evolution.

Having verified the consistency of the diffeomorphism constraint under time evolution, we now move on to the Gauss constraint. Application of the Leibniz rule to the first equation of (35) yields

$$\dot{G}_a = \dot{B}_e^i D_i \Psi_{ae} + B_e^i D_i \dot{\Psi}_{ae} + B_e^i (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) \dot{A}_i^b. \quad (48)$$

Upon substitution of (30) and (34) into (48), we have

$$\begin{aligned} \dot{G}_a = (-\delta_{\bar{\theta}} B_e^i - i \epsilon^{ijk} D_j H_k^e) D_i \Psi_{ae} + \\ + B_e^m D_m \left[-\delta_{\bar{\theta}} \Psi_{ae} - i \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) H_k^f \right] + \\ + B_e^i (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) (-\delta_{\bar{\theta}} A_i^b - i H_i^b). \end{aligned} \quad (49)$$

Using the Leibniz rule to re-combine the $\delta_{\bar{\theta}}$ terms of (49), we have

$$\begin{aligned} \dot{G}_a = -\delta_{\bar{\theta}} G_a - i \epsilon^{ijk} \left\{ (D_j H_k^e) D_i \Psi_{ae} + \right. \\ \left. + B_e^m D_m \left[(B^{-1})_i^e (D_j \Psi_{af}) H_k^f \right] \right\} - i (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) B_e^i H_i^b. \end{aligned} \quad (50)$$

The requirement of consistency is that we must show that the right hand side of (50) vanishes weakly. First, we will show that the third term on the right hand side of (50) vanishes up to terms of linear order and higher in the diffeomorphism constraint. This term, up to an

insignificant numerical factor, has two contributions. The first contribution is

$$\begin{aligned} & (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag})B_e^i(H_{(1)})_i^b = \\ & = \sqrt{-g}(f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag})(\Psi^{-1}\Psi^{-1})^{eb} = \\ & = \sqrt{-g}\left[f_{abf}(\Psi^{-1})^{fb} + f_{ebg}(\Psi^{-1}\Psi^{-1})^{eb}\Psi_{ag}\right] \sim \delta_a^{(1)}(\vec{\psi}) \sim 0, \end{aligned} \quad (51)$$

which is directly proportional to a nonlinear function of first order in ψ_d which is proportional to the diffeomorphism constraint. The second contribution to the third term on the right hand side of (50) is

$$\begin{aligned} & (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag})B_e^i(H_{(2)})_i^b = \\ & = (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag})\epsilon_{kmn}N^k B_e^m B_b^n = \\ & = (f_{abf}\Psi_{fe} + f_{ebg}\Psi_{ag})(\det\|B\|)N^k(B^{-1})_k^d\epsilon_{deb}. \end{aligned} \quad (52)$$

We can now apply the epsilon identity (40) to (52), using the fact that $f_{abc} = \epsilon_{abc}$. This yields

$$\begin{aligned} & (\det\|B\|)N^k(B^{-1})_k^d\left[(\delta_{fd}\delta_{ae} - \delta_{fe}\delta_{ad})\Psi_{fe} + 2\delta_{dg}\Psi_{ag}\right] = \\ & = (\det\|B\|)N^k(B^{-1})_k^d(\Psi_{da} - \delta_{ad}\text{tr}\Psi + 2\Psi_{ad}) \equiv \delta_a^{(2)}(\vec{N}), \end{aligned} \quad (53)$$

which does not vanish, and neither is it expressible as a constraint. For the Gauss' law constraint to be consistent under time evolution, a necessary condition is that this $\delta_a^{(2)}(\vec{N})$ term must be exactly cancelled by another term arising from the variation.

Let us expand the terms in (50) associated with the square brackets. This is given, applying the Leibniz rule to the second term, by

$$\begin{aligned} & \epsilon^{ijk}(D_j H_k^e)(D_i \Psi_{ae}) + \epsilon^{ijk}B_e^m D_m \left[(B^{-1})_i^e (D_j \Psi_{ae}) H_k^f \right] = \\ & = \epsilon^{ijk}(D_j H_k^e)(D_i \Psi_{ae}) - \epsilon^{ijk}B_e^m (B^{-1})_n^e (D_m B_g^n)(B^{-1})_i^g \times \\ & \times (D_j \Psi_{af}) H_k^f + \epsilon^{mjk}(D_m D_j \Psi_{af}) H_k^f + \epsilon^{mjk}(D_j \Psi_{af})(D_m H_k^f). \end{aligned} \quad (54)$$

The first and last terms on the right hand side of (54) cancel, which can be seen by relabelling of indices. Upon application of the definition of curvature as the commutator of covariant derivatives to the third term, then (54) reduces to

$$- \epsilon^{ijk}(D_n B_g^n)(B^{-1})_i^g (D_j \Psi_{af}) H_k^f + H_k^f B_b^k (f_{abc}\Psi_{cf} + f_{fbc}\Psi_{ac}). \quad (55)$$

The first term of (55) vanishes on account of the Bianchi identity and

the second term contains two contributions which we must evaluate. The first contribution is given by

$$\begin{aligned} (H_{(2)})^f_k B_b^k (f_{abc} \Psi_{cf} + f_{fbc} \Psi_{ac}) &= \\ &= (\det \|B\|) N^k (B^{-1})^d_k \epsilon_{dbf} (f_{abc} \Psi_{cf} + f_{fbc} \Psi_{ac}). \end{aligned} \quad (56)$$

Applying (40), then (56) simplifies to

$$\begin{aligned} (\det \|B\|) N^k (B^{-1})^d_k \left[(\delta_{da} \delta_{fc} - \delta_{dc} \delta_{fa}) \Psi_{cf} - 2 \delta_{dc} \Psi_{ac} \right] &= \\ = (\det \|B\|) N^k (B^{-1})^d_k (\delta_{da} \text{tr} \Psi - \Psi_{da} - 2 \Psi_{ad}) &= -\delta_a^{(2)}(\vec{N}), \end{aligned} \quad (57)$$

with $\delta_a^{(2)}(\vec{N})$ as given in (51). So putting the results of (54), (55) and (57) into (50), we have

$$\begin{aligned} \dot{G}_a &= -\delta_{\vec{\theta}} G_a + \delta_a^{(2)}(\vec{N}) + \delta_a^{(1)}(\vec{\psi}) + \delta_a^{(1)}(\vec{\psi}) - \delta_a^{(2)}(\vec{N}) = \\ &= -\delta_{\vec{\theta}} G_a + 2 \delta^{(1)}(\vec{\psi}), \end{aligned} \quad (58)$$

whence the $\delta^{(2)}(\vec{\psi})$ terms have cancelled out. The velocity of the Gauss' law constraint is a linear combination of the Gauss constraint with terms of the diffeomorphism constraint of linear order and higher. Hence the time evolution of the Gauss' law constraint is consistent in the sense that we have defined, since $\delta^{(1)}(\vec{\psi})$ vanishes for $\psi_d = 0$.

§7. Consistency of the Hamiltonian constraint under time evolution. The time derivative of the Hamiltonian constraint, the third equation of (35), is given by

$$\dot{H} = \left[\frac{d}{dt} \left(\sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} \right) \right] (\Lambda + \text{tr} \Psi^{-1}) + \frac{\sqrt{-g}}{N} \frac{d}{dt} (\Lambda + \text{tr} \Psi^{-1}) \quad (59)$$

which has split up into two terms. The first term is directly proportional to the Hamiltonian constraint, therefore it is already consistent. We will nevertheless expand it using (30) and (34)

$$\begin{aligned} \frac{1}{2} \left[(B^{-1})^d_i \dot{B}_d^i + (\Psi^{-1})^{ae} \dot{\Psi}_{ae} \right] \sqrt{\det \|B\|} \sqrt{\det \|\Psi\|} (\Lambda + \text{tr} \Psi^{-1}) &= \\ = \frac{1}{2} \left\{ (B^{-1})^d_i (-\delta_{\vec{\theta}} B_d^i - i \epsilon^{ijk} D_j H_k^d) + \right. & \\ \left. + (\Psi^{-1})^{ae} \left[-\delta_{\vec{\theta}} \Psi_{ae} - i \epsilon^{ijk} (B^{-1})^e_i (D_j \Psi_{af}) \right] H_k^f \right\} H. \end{aligned} \quad (60)$$

We will be content to compute the $\delta_{\vec{\theta}}$ terms of (60). These are

$$(B^{-1})^d_i \delta_{\vec{\theta}} B_d^i = (B^{-1})^d_i f_{dbf} B_b^i A_0^f = \delta_{db} f_{dbf} A_0^f = 0 \quad (61)$$

on account of antisymmetry of the structure constants, and

$$(\Psi^{-1})^{ea} \delta_{\vec{\theta}} \Psi_{ae} = (\Psi^{-1})^{ea} (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) A_0^b = 0, \quad (62)$$

also due to antisymmetry of the structure constants. We have shown that the first term on the right hand side of (59) is consistent with respect to time evolution. To verify consistency of the Hamiltonian constraint under time evolution, it remains to show that the second term is weakly equal to zero. It suffices to show this just for the second term, in brackets, of (59)

$$\begin{aligned} \frac{d}{dt} (\Lambda + \text{tr} \Psi^{-1}) &= -(\Psi^{-1} \Psi^{-1})^{fe} \dot{\Psi}_{ef} = \\ &= (\Psi^{-1} \Psi^{-1})^{ae} \left[\delta_{\vec{\theta}} \Psi_{ae} - i \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) H_k^f \right], \end{aligned} \quad (63)$$

where we have used (34). Equation (63) has split up into two terms, of which the first term is

$$\begin{aligned} (\Psi^{-1} \Psi^{-1})^{ea} \delta_{\vec{\theta}} \Psi_{ae} &= (\Psi^{-1} \Psi^{-1})^{ea} (f_{abf} \Psi_{fe} + f_{ebg} \Psi_{ag}) A_0^b = \\ &= \left[f_{abf} (\Psi^{-1})^{fa} + f_{ebg} (\Psi^{-1})^{eg} \right] A_0^b = m(\vec{\psi}) \sim 0 \end{aligned} \quad (64)$$

which vanishes weakly since it is a nonlinear function of at least linear order in ψ_d . The second term of (63) splits into two terms which we must evaluate. The first contribution is proportional to

$$\begin{aligned} (\Psi^{-1} \Psi^{-1})^{ea} \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) (H_{(1)})_k^f &= \\ = \sqrt{-g} (\Psi^{-1} \Psi^{-1})^{ea} \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) (B^{-1})_k^d (\Psi^{-1} \Psi^{-1})^{df}. \end{aligned} \quad (65)$$

Proceeding from (65) and using (43) to simplify the magnetic field contributions, we have

$$\begin{aligned} -\sqrt{-g} (\Psi^{-1} \Psi^{-1})^{ea} (\Psi^{-1} \Psi^{-1})^{df} (\det \|B\|)^{-1} \epsilon^{edg} B_g^j D_j \Psi_{af} &= \\ = -\sqrt{-g} (\det \|B\|)^{-1} \epsilon^{edg} (\Psi^{-1} \Psi^{-1})^{ea} (\Psi^{-1} \Psi^{-1})^{df} \times \\ \times \mathbf{v}_g \{ \Psi_{af} \} \equiv \mathbf{v} \{ \vec{\psi} \} \end{aligned} \quad (66)$$

for some vector field \mathbf{v} . We have used the fact that the term in (66) quartic in Ψ^{-1} is antisymmetric in a and f due to the epsilon symbol. Hence Ψ_{af} as acted upon by \mathbf{v}_g can appear only in an antisymmetric combination, and is therefore proportional to the diffeomorphism constraint ψ_d whose spatial derivatives weakly vanish. Therefore (66) presents a consistent contribution to the time evolution of H , which leaves remaining the second contribution to the second term of (63). This term is propor-

tional to

$$\begin{aligned}
& (\Psi^{-1}\Psi^{-1})^{ea} \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) (H_{(2)})_k^f = \\
& = (\Psi^{-1}\Psi^{-1})^{ea} \epsilon^{ijk} (B^{-1})_i^e (D_j \Psi_{af}) \epsilon_{mnk} N^m B_f^n = \\
& = (\delta_m^i \delta_n^j - \delta_n^i \delta_m^j) (B^{-1})_i^e N^m B_f^n (\Psi^{-1}\Psi^{-1})^{ea} (D_j \Psi_{af}) \quad (67)
\end{aligned}$$

where we have applied the epsilon identity. Proceeding from the right hand side of (67), we have

$$\begin{aligned}
& \left[N^i (B^{-1})_i^e B_f^j - \delta_{ef} N^j \right] (\Psi^{-1}\Psi^{-1})^{ea} (D_j \Psi_{af}) = \\
& = (-g)^{-1/2} N^i H_i^a \mathbf{v}_f \{ \Psi_{af} \} - (\Psi^{-1}\Psi^{-1})^{fa} (N^j D_j \Psi_{af}) = \\
& = (-1)^{-1/2} N^i H_i^a G_a - N^j D_j (\Lambda + \text{tr} \Psi^{-1}). \quad (68)
\end{aligned}$$

The first term on the final right hand side of (68) is proportional to the Gauss' law constraint, and the second term is proportional to the derivative of the Hamiltonian constraint, since $N^i D_i = N^i \partial_i$ on scalars. To obtain this second term we have added in Λ which becomes annihilated by ∂_j . Substituting (64), (66) and (68) into (63), then we have

$$\dot{H} = \sim \hat{O}(\vec{\psi}) + (-g)^{-1/2} N^i H_i^a G_a + \hat{T} [(-g)^{-1/2} H], \quad (69)$$

where \hat{O} and \hat{T} are operators consisting of spatial derivatives acting to the right and c numbers. The time derivative of the Hamiltonian constraint is a linear combination of the Gauss' law and Hamiltonian constraints and its spatial derivatives, plus terms of linear order and higher in the diffeomorphism constraint and its spatial derivatives. Hence we have shown that the Hamiltonian constraint is consistent under time evolution.

§8. Recapitulation and discussion. The most important aspect of consistency required for any totally constrained system is that the constraint surface be preserved under time evolution for all time. If upon taking the time derivative of a constraint one obtains a quantity which does not vanish on-shell, then this introduces additional constraints on the system which must similarly be verified to be consistent with the existing constraints. One must proceed in this manner until a self-consistent system of constraints is obtained. Hopefully, one is then left with a system which still contains nontrivial dynamics. In the case of the instanton representation, we have performed this test on all of the constraints arising from the action.

The final equations governing the time evolution of the initial value

constraints are given weakly by

$$\left. \begin{aligned} \dot{\psi}_d &= \left[i (\det \|B\|)^{-1} (\Psi^{-1} \Psi^{-1})^{da} + \epsilon_{dae} N^i (B^{-1})_i^e \right] G_a + \\ &\quad + (A_0^b \epsilon_{bdh} - \delta_{dh} N^j D_j) \psi_h + i (\det \|B\|)^{-1} \mathbf{v}_d \{ \Lambda + \text{tr } \Psi^{-1} \} \\ \dot{G}_a &= -f_{abc} A_0^b G_c + \delta_a^{(1)}(\vec{\psi}) \\ \dot{H} &= \left[-\frac{i}{2} \epsilon^{ijk} (B^{-1})_i^d (D_j H_k^d) + \right. \\ &\quad \left. + \epsilon^{ijk} (B^{-1})_i^e (\Psi^{-1})^{ae} (D_j \Psi_{af}) H_k^f - N^j \partial_j \right] (\Lambda + \text{tr } \Psi^{-1}) + \\ &\quad + (-g)^{-1/2} N^i H_i^a G_a - \sqrt{-g} (\det \|B\|)^{-1} \times \\ &\quad \times \epsilon^{edg} (\Psi^{-2} \Psi^{-1})^{ea} (\Psi^{-1} \Psi^{-1})^{df} \mathbf{v}_g \{ \epsilon_{afh} \psi_h \} + m(\vec{\psi}) \end{aligned} \right\}. \quad (70)$$

Equations (70) show that all constraints derivable from the the action (10) are preserved under time evolution, since their time derivatives yield linear combinations of the same set of constraints and their spatial derivatives, with no additional constraints. In spite of the fact that we have defined neither the canonical structure of (1) nor any Poisson brackets, this is tantamount to the Dirac consistency of (1).

Equations (70) can be written schematically in the following form

$$\left. \begin{aligned} \dot{\vec{H}} &\sim \vec{H} + \vec{G} + H \\ \dot{\vec{G}} &\sim \vec{G} + \Phi(\vec{H}) \\ \dot{H} &\sim H + \vec{G} + \Phi(\vec{H}) \end{aligned} \right\}, \quad (71)$$

where Φ is some nonlinear function of the diffeomorphism constraint \vec{H} , which is of at least first order in \vec{H} . In the Hamiltonian formulation of a theory, one identifies time derivatives of a variable f with $\dot{f} = \{f, \mathbf{H}\}$, the Poisson brackets of f with a Hamiltonian \mathbf{H} . So while we have not defined Poisson brackets, equation (71) implies the existence of Poisson brackets associated to some Hamiltonian \mathbf{H}_{Inst} for the action (10), with

$$\left. \begin{aligned} \{\vec{H}, \mathbf{H}_{\text{Inst}}\} &\sim \vec{H} + \vec{G} + H \\ \{\vec{G}, \mathbf{H}_{\text{Inst}}\} &\sim \vec{G} + \Phi(\vec{H}) \\ \{H, \mathbf{H}_{\text{Inst}}\} &\sim H + \Phi(\vec{H}) + \vec{G} \end{aligned} \right\}. \quad (72)$$

So the main result of this paper has been to demonstrate that the instanton representation of Plebanski gravity forms a consistent system,

in the sense that the constraint surface is preserved under time evolution. As a direction of future research we will compute the algebra of constraints for (10) directly from its Poisson brackets. Nevertheless it will be useful for the present paper to think of equations (70) in the Dirac context, mainly for comparison with alternate formulations of General Relativity. This will bring us back to the Ashtekar variables.

Let us revisit (9) for each constraint with the total Hamiltonian \mathbf{H}_{Ash} and compare with (72). This is given schematically by

$$\left. \begin{aligned} \{\vec{H}, \mathbf{H}_{\text{Ash}}\} &\sim \vec{H} + \vec{G} + H \\ \{\vec{G}, \mathbf{H}_{\text{Ash}}\} &\sim \vec{G} + \vec{H} \\ \{H, \mathbf{H}_{\text{Ash}}\} &\sim H + \vec{H} \end{aligned} \right\}. \quad (73)$$

Comparison of (73) with (72) shows an essentially similar structure for the top two lines involving \vec{H} and \vec{G} (we regard the linearity versus nonlinearity of the diffeomorphism constraints on the right hand side as a dissimilarity, albeit a minor dissimilarity). But there is a marked dissimilarity with respect to the Hamiltonian constraint H . Note that there is a Gauss' law constraint appearing in the right hand side of the last line of (72) whereas there is no such constraint on the corresponding right hand side of (73). This means that while the Hamiltonian constraint is gauge-invariant under $\text{SO}(3, \mathbb{C})$ gauge-transformations as implied by (9) and (73), this is not the case in (72). This means that the action (10), which as shown in [1] describes General Relativity for Petrov Types I, D and O, suggests a different role for the Gauss' law and Hamiltonian constraints than the action (5), which also describes General Relativity. The conclusion is therefore that I_{Inst} and I_{Ash} at some level must correspond to genuinely different descriptions of General Relativity, a feature which would have been missed had we applied the step-by-step Dirac procedure.*

On a final note, there is a common misconception that I_{Inst} is the same action as a certain action leading to the CDJ pure spin connection formulation of [5], or should fall under the CDJ formalism. Additionally, we would like to dispell any notion that the pure spin connection action $I_{\text{CDJ}} = I_{\text{CDJ}}[\eta, A]$ or its antecedent $I_1 = I_1[\Psi, A]$ are directly related to I_{Inst} . They are related in the sense that $I_{\text{CDJ}}, I_1 \subset I_{\text{Inst}}$, but the converse is not true for the reasons shown in [1], which we will not repeat here.

*The Dirac procedure naively applied to I_{Inst} would lead one directly to I_{Ash} via (4), which might suggest superficially that these two theories are the same. However, as the results of this paper show, I_{Inst} is a stand-alone action with an algebraic structure different from I_{Ash} .

The Ashtekar action I_{Ash} has been shown in [8] to be the 3 + 1 decomposition of I_{CDJ} for Petrov Types I, D and O. We have shown in §2 that I_{Inst} as well exhibits this feature. However, this is not the case on the noncanonical phase space $\Omega_{\text{Inst}} = (\Psi_{ae}, A_i^a)$, which the present paper has demonstrated.

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